Convexity of the zeros of some orthogonal polynomials and related functions

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Abstract

We study convexity properties of the zeros of some special functions that follow from the convexity theorem of Sturm. We prove results on the intervals of convexity for the zeros of Laguerre, Jacobi and ultraspherical polynomials, as well as functions related to them, using transformations under which the zeros remain unchanged. We give upper as well as lower bounds for the distance between consecutive zeros in several cases.

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1 Introduction

The Sturm comparison theorem for solutions of second order differential equations of the form $y'' + F(t)y = 0$ (cf. [14]) has been significantly extended since its publication 170 years ago. Some of the immediate applications to zeros of the solutions $y(t)$, and those of the derivative $y'(t)$, include Sonin's theorem on the monotonicity of extrema of such solutions (cf. [11]), and a result known as Sturm's convexity theorem, first mentioned in [14], on the monotonicity of distances between the zeros of the solution (cf. [9], [11] and [15]).

Sonin's theorem was extended to more general differential equations of the form $P(t)y'' + Q(t)y' + y = 0$ using a remarkably simple proof (cf. [13], p.443). In this form the theorem can be directly applied to classical orthogonal polynomials such as Hermite, Laguerre and Jacobi polynomials, providing the monotonicity of their relative maximum values and estimates on their supremum norm. This has been done from a different perspective in [8], recovering results for Legendre, Laguerre and Jacobi polynomials given in [16].

In this paper, we consider the implications of the convexity theorem of Sturm for the convexity of the zeros and bounds on the distance between the zeros of some classical orthogonal polynomials and functions related to them.

2 Convexity and spacing of zeros

The convexity theorem and an obvious consequence of the comparison theorem of Sturm, already noted in [14], can be summarised as follows.

Theorem 2.1 *[3] Let $y''(t) + F(t)y(t) = 0$ be a second-order differential equation in normal form, where F is continuous in (a, b). Let $y(t)$ be a nontrivial solution in (a, b), and let $x_1 < \ldots < x_k < x_{k+1} < \ldots$ denote the consecutive zeros of $y(t)$ in (a, b). Then*

1. *if $F(t)$ is strictly increasing in (a, b), $x_{k+2} - x_{k+1} < x_{k+1} - x_k$,*

2. *if $F(t)$ is strictly decreasing in (a, b), $x_{k+2} - x_{k+1} > x_{k+1} - x_k$.*

3. *if there exists $M > 0$ such that $F(t) < M$ in (a, b) then*

$$\Delta x_k \equiv x_{k+1} - x_k > \frac{\pi}{\sqrt{M}},$$

4. *if there exists $m > 0$ such that $F(t) > m$ in (a, b) then*

$$\Delta x_k < \frac{\pi}{\sqrt{m}},$$

We say that the zeros of y are concave (convex) on (a, b) for the first (second) case.

The convexity theorem has been used to obtain the variation of convexity properties with respect to a parameter, or the order, for the zeros of gamma, q-gamma, Bessel, cylindrical and Hermite functions as described in the survey paper [12].

In order to apply the convexity theorem to special functions that are solutions of second-order differential equations, the differential equation has to be transformed into normal form. One simple way to do this is through the following change of dependent variable. Let

$$x'' + g(t)x' + f(t)x = 0$$

be a second-order differential equation and set

$$y = x \exp\left(\frac{1}{2}\int^t g(s)ds\right) \tag{1}$$

The corresponding equation for y is in normal form:

$$y'' + F(t)y = 0,$$

where $F(t) = f(t) - \frac{1}{4}g^2(t) - \frac{1}{2}g'(t)$. The advantage of this transformation is that it does not change the independent variable, and the zeros of x and y are the same. Hille [7] already used transformation (1) to prove the convexity of zeros of the Hermite polynomials.

It is also possible to consider other changes of variable and obtain information on the convexity of the transformed zeros. This was done already by Szegő for the ultraspherical polynomials [16, Theorem 6.3.3] and lately by Deaño, Gil and Segura [3, 4] for hypergeometric functions.

We will consider the convexity and spacing of the zeros of special functions such as Laguerre, Jacobi and, as a special case, the ultraspherical polynomials, for fixed order n, by transforming their differential equations to normal form using (1). Sturm [14] used the same method to obtain results on the convexity and spacing of the zeros of the Bessel function. Interesting work on the spacing of the zeros of Jacobi polynomials, as the degree changes, is done in [1]. For higher monotonicity refer to, amongst others, [10] and [5].

We note that since the convexity theorem is applicable to any oscillating solutions of second order differential equations in normal form, the results we obtain are not restricted to the polynomial cases, i.e. n need not necessarily be an integer, as long as the corresponding functions are oscillating on the interval under consideration. In addition, the results can be extended to

parameter values where the polynomials are no longer orthogonal, since quasi-orthogonality ensures the existence of some zeros on the interval of orthogonality (cf. [2]).

3 Laguerre polynomials

The differential equation

$$tx'' + (\alpha + 1 - t)x' + nx = 0$$

satisfied by the Laguerre polynomials, $L_n^\alpha(t)$, orthogonal on $(0, \infty)$ with respect to the weight function $t^\alpha e^{-t}$ when $\alpha > -1$, is transformed to

$$y'' + F(t)y \;=\; 0$$

by (1) where

$$F(t) \;=\; \frac{-t^2 + 2\alpha t + 2t + 4nt - \alpha^2 + 1}{4t^2}. \tag{2}$$

$F'(t)$ changes sign at

$$t_0 := \frac{\alpha^2 - 1}{\alpha + 2n + 1}. \tag{3}$$

Theorem 3.1 *The zeros of $L_n^\alpha(t)$ on $(0, \infty)$ are*

1. *all convex if $n > 0$ and $-1 < \alpha \le 3$*

2. *all convex if $\alpha > 3$ and $0 < n < \frac{\alpha+1}{\alpha-3}$*

3. *concave for $t < t_0$ and convex for $t > t_0$ when $\alpha > 3$, $n > \frac{\alpha+1}{\alpha-3}$ and t_0 is defined by (3).*

Moreover, for the distance between consecutive zeros we have the general estimate

$$\Delta x_k > \frac{\pi\sqrt{2}}{\sqrt{2\alpha n + \alpha + 2n^2 + 2n + 1}} \qquad k = 1, \ldots, n-1 \tag{4}$$

and also if $x_k > t_0$ then

$$\Delta x_k > \frac{\pi}{\sqrt{F(x_k)}} \qquad k = 1, \ldots, n-1 \tag{5}$$

and

$$\Delta x_k < \frac{\pi}{\sqrt{F(x_{k+1})}} \qquad k = 1, \ldots, n-2 \tag{6}$$

where F is defined by (2).

Proof. For $|\alpha| < 1$, $t_0 < 0$, hence $F(t)$ will be decreasing on $(0, \infty)$. When $\alpha \geq 1$, $F(t)$ is increasing on $(0, t_0)$ and decreasing on (t_0, ∞). Let x_1 denote the smallest zero of L_n^α, then we know that $x_1 > \frac{\alpha+1}{n}$ (cf. [6]). This implies that when $t_0 < \frac{\alpha+1}{n}$, $F(t)$ will be decreasing on the interval (x_1, ∞). An easy calculation shows that this condition is equivalent to either $\alpha \leq 3$ or $\alpha > 3$ and $n < \frac{\alpha+1}{\alpha-3}$. The estimates on the distance Δx_k follow from Theorem 2.1(3),(4). The maximum of F is at t_0 and $F(t_0) > 0$, therefore we can take $F(t_0)$ as M to obtain (4). For (5) and (6), we use the fact that when $x_k > t_0$, F is monotone decreasing on (x_k, x_{k+1}). In fact, F is monotone decreasing on $(0, \infty)$ and tends to -1/4 as $t \to \infty$, so there is exactly one point t_1 on (t_0, ∞), where F crosses the x-axis. The form of the differential equation implies that if $F(t) < 0$ and $y(t) > 0$, the graph will be concave up and similarly, if $y(t) < 0$, the graph will be concave down. Hence there can be at most one zero of the Laguerre polynomial to the right of t_1. This means that $F(x_{n-1})$ is positive, but $F(x_n)$ may be negative and therefore the index in (6) only runs up to $n - 2$. ∎

Remark 3.2 *An interesting question is whether it is possible to find α and n values so that the first several zeros of the Laguerre polynomial are concave. This would require t_0 to be greater than x_1. In this regard we note that t_0 is always less than $\frac{(\alpha+1)(\alpha+2)}{\alpha+n+1}$, which is the upper bound for x_1 given in [6]. It is also always less than $\frac{(\alpha+1)(\alpha+3)}{\alpha+2n+1}$, the upper bound given in [16].*

4   Jacobi polynomials

The differential equation for Jacobi polynomials, $P_n^{(\alpha,\beta)}(t)$, orthogonal on $(-1, 1)$ with respect to the weight function $(1-t)^\alpha(1+t)^\beta$ when α, $\beta > -1$, is

$$(1 - t^2)x''(t) + (\beta - \alpha - (\alpha + \beta + 2)t)x'(t) + n(n + \alpha + \beta + 1)x(t) = 0.$$

In the normal form, $y'' + F(t)y = 0$, we have

$$F(t) \;=\; \frac{-zt^2 - 2(x - y)t - 2x - 2y + z}{4(t^2 - 1)^2} \tag{7}$$

$$\text{with } x \;=\; \alpha^2 - 1$$
$$y \;=\; \beta^2 - 1$$
$$z \;=\; (\alpha + \beta + 2n)(\alpha + \beta + 2n + 2).$$

Also

$$F'(t) = \frac{zt^3 + 3(x - y)t^2 + (4x + 4y - z)t + (x - y)}{2(t^2 - 1)^3} := \frac{j(t)}{2(t^2 - 1)^3}$$

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and we denote the discriminant of $j'(t)$ by

$$D := 12(3x^2 + 3y^2 + z^2 - 6xy - 4xz - 4yz).$$

For the convexity theorem to be applicable, we need oscillating solutions. The condition on the parameters for this is (cf. [3])

$$n > 0, \ n + \alpha + \beta > 0, \ n + \alpha > 0, \ n + \beta > 0.$$

From now on we shall assume that the coefficients satisfy these conditions.

Theorem 4.1 *If* $|\alpha| > 1$, $|\beta| < 1$ *and* $D < 0$, *all the zeros of* $P_n^{(\alpha,\beta)}$ *on the interval* $(-1, 1)$ *are convex.*

Proof. $F(t)$ is a rational function with vertical asymptotes at $t = \pm 1$. If $|\alpha| > 1$ and $|\beta| < 1$, then $j(-1) = -8y > 0$ and $j(1) = 8x > 0$, so that $\lim_{t \to -1} F(t) = \infty$ and $\lim_{t \to 1} F(t) = -\infty$. $D < 0$ implies that $j'(t) \neq 0$ for $t \in (-1, 1)$ and hence $j(t)$ will have no extreme values on this interval. It follows that $F(t)$ is monotone decreasing on $(-1, 1)$ and Theorem 2.1(2) yields the result. ∎

Note that the conditions of Theorem 4.1 are satisfied if, for example, y is sufficiently small and $x < z < 3x$. This is true if, for instance, β is sufficiently close to -1, $\alpha > 1$, and $n < \frac{1}{2}(-\alpha + \sqrt{3\alpha^2 - 2})$. Also $D < 0$ will be satisfied for sufficiently large α if we fix β and n. However, for fixed α and β the discriminant D is positive for large n.

More results of this type can be obtained by ensuring the positivity of $j(t)$ on $(-1, 1)$. Denote the zeros of $j'(t)$ by

$$t_{1,2} = \frac{6(y - x) \pm \sqrt{D}}{6z},$$

then we have that the zeros of $P_n^{(\alpha,\beta)}$ on $(-1, \ 1)$ are convex if $|\alpha| > 1$ and $|\beta| < 1$ and if $t_i \in (-1, 1)$ for some $i = 1, 2$ then $j(t_i) > 0$. One can prove conditions for concavity of the zeros of $P_n^{(\alpha,\beta)}$ on the whole interval $(-1, \ 1)$ in a similar manner.

The general study of the convexity of the zeros is difficult, since the convexity intervals are determined by the roots of $j(t)$, which are hard to handle due to the 3 parameters. However, there are still some things that can be said about the general case.

The oscillation condition $z > 0$ implies that j has a concave part, followed by a convex, the inflection point being at $t_0 := \frac{y-x}{z}$. Therefore it follows from the shape of $j(t)$ that there may be up to 4 different intervals of changing concavity for the zeros on $(-1, \ 1)$: (from left to right) concave-convex-concave-convex. Any of these intervals may be missing from the

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sequence, for example, concave-convex-concave or convex-concave are possible for certain parameter values.

It is interesting to analyse the convexity of the zeros of $P_n^{(\alpha,\beta)}$ for sufficiently large degree.

Theorem 4.2 *Let α and β be fixed and let $n \to \infty$, then the convexity of the zeros of $P_n^{(\alpha,\beta)}$ on $(-1, 1)$ changes in the following way (from left to right):*

1. *if $|\alpha| \leq 1$ and $|\beta| \leq 1$ then convex-concave,*

2. *if $|\alpha| \leq 1$ and $|\beta| > 1$ then concave-convex-concave,*

3. *if $|\alpha| > 1$ and $|\beta| \leq 1$ then convex-concave-convex,*

4. *if $|\alpha| > 1$ and $|\beta| > 1$ then concave-convex-concave-convex.*

Proof. If α and β are fixed and $n \to \infty$, an easy calculation shows that the local extremum locations of $j(t)$ tend to $\pm\sqrt{3}/3$. Since $z > 0$, the local extremum near $t = -\sqrt{3}/3$ will be the maximum and this maximum value tends to ∞. Similarly, the minimum value near $\sqrt{3}/3$ tends to $-\infty$. Since the inflection point $t_0 = \frac{y-x}{z} \to 0$, there is at least one change of concavity in $(-1, 1)$ (from convex to concave) and whether there are more, depends on the sign of $j(-1)$ and $j(1)$. Now $j(-1) = 8(1 - \beta^2)$ and $j(1) = 8(\alpha^2 - 1)$ and the result follows. ∎

5 Ultraspherical polynomials

An important special case of the Jacobi polynomials are the ultraspherical polynomials, $P_n^{(\alpha,\alpha)}(t)$ where $\alpha = \beta$. In this case

$$F(t) = \frac{-(\alpha + n)(\alpha + n + 1)t^2 + (1 + n + n^2 + \alpha + 2\alpha n)}{(t^2 - 1)^2},$$

the numerator of $F'(t)$ is $j(t) = 4[(\alpha + n)(\alpha + n + 1)t^3 - (2 + n + n^2 + \alpha + 2\alpha n - \alpha^2)t]$ and the discriminant of $j'(t)$ is $D = 192(\alpha + n)(\alpha + n + 1)(2 + n + n^2 + \alpha + 2\alpha n - \alpha^2)$. Note that the leading coefficient of $j(t)$ is positive when $\alpha > -1$. The point of inflection of $j(t)$ is $t_0 = 0$ and hence the convexity of zeros changes exactly in the middle of the interval $(-1, 1)$. The local extrema of $j(t)$ are at

$$t_{1,2} = \pm\sqrt{\frac{(n + \alpha)(n + \alpha + 1) - 2(\alpha^2 - 1)}{3(n + \alpha)(n + \alpha + 1)}},$$

and the two remaining zeros of $j(t)$ are

$$T_{1,2} = \pm\sqrt{\frac{(n + \alpha)(n + \alpha + 1) - 2(\alpha^2 - 1)}{(n + \alpha)(n + \alpha + 1)}} = t_{1,2}\sqrt{3}. \tag{8}$$

where T_1 denotes the negative and T_2 the positive zero.

Theorem 5.1 *If $|\alpha| \leq 1$, the zeros of $P_n^{(\alpha,\alpha)}$ on $(-1,\ 0)$ are convex and those on $(0,\ 1)$ are concave. In addition*

$$\Delta x_k < \frac{\pi}{\sqrt{F(0)}} = \frac{\pi}{\sqrt{2\alpha n + \alpha + n^2 + n + 1}},$$

and for the positive zeros we have

$$\frac{\pi}{\sqrt{F(x_{k+1})}} < \Delta x_k < \frac{\pi}{\sqrt{F(x_k)}}.$$

Proof. For $|\alpha| \leq 1$ we have $T_1 < -1$ and $T_2 > 1$, so j is positive on $(-1,\ 0)$ and negative on $(0,\ 1)$. Therefore $F(t)$ is decreasing on $(-1,0)$ and increasing on $(0,1)$ and the convexity of the zeros follows from Theorem 2.1(1),(2). In addition, $F(0) > 0$ is a minimum value, so we have an upper bound on the distance between any two consecutive zeros from Theorem 2.1(4). Finally, since $F(t)$ is increasing on $(0,1)$, $0 < F(x_i) < F(x) < F(x_{i+1})$ for $x \in (x_i, x_{i+1})$, where $x_i\ x_{i+1}$ are any two consecutive positive zeros and the last inequality follows from Theorem 2.1(3),(4). ∎

Note that this result also applies to Chebyshev and Legendre polynomials as special cases with $\alpha = \frac{1}{2}$ and $\alpha = 0$ respectively.

Theorem 5.2 *Let $|\alpha| > 1$ and $(n+\alpha)(n+\alpha+1) \leq 2(\alpha^2-1)$ then the zeros of $P_n^{(\alpha,\alpha)}$ on $(-1,\ 0)$ are concave and those on $(0,\ 1)$ are convex. Furthermore*

$$\Delta x_k > \frac{\pi}{\sqrt{F(0)}}.$$

Proof. If $|\alpha| > 1$, and $D < 0$, j has no local extremum and is monotone increasing on $(-1,1)$. This, together with the fact that $j(0) = 0$, implies that $F(t)$ is increasing on $(-1,0)$ and decreasing on $(0,1)$, and the result follows. ∎

Note that we cannot obtain estimates involving x_k, x_{k+1}, because $F \to -\infty$ as $t \to \pm 1$.

Theorem 5.3 *Let $|\alpha| > 1$ and $(n+\alpha)(n+\alpha+1) > 2(\alpha^2-1)$ then the zeros of $P_n^{(\alpha,\alpha)}$ are concave on $(-1,\ T_1)$ and $(0,\ T_2)$ and convex on $(T_1,\ 0)$ and $(T_2,\ 1)$, where $T_{1,2}$ are as in (8). We also have that*

$$\Delta x_k > \frac{\pi}{\sqrt{F(T_2)}},$$

moreover, if $(x_k,\ x_{k+1}) \subset (T_1,\ T_2)$ then

$$\Delta x_k < \frac{\pi}{\sqrt{F(0)}},$$

and if $(x_k,\ x_{k+1}) \subset (0,\ T_2)$, then

$$\frac{\pi}{\sqrt{F(x_{k+1})}} < \Delta x_k < \frac{\pi}{\sqrt{F(x_k)}}.$$

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Proof. When $|\alpha| > 1$ and $D > 0$, $j(t)$ has 3 zeros on $(-1, 1)$, namely at T_1, 0 and T_2 with $F(t)$ having local maxima at $T_{1,2}$ and a local minimum at $t = 0$. ∎

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